<u>Magic Numberings for Polyhedra</u>

Do you like brain teasers with numerical aspects involved ? Then try to number a cube's edges in such a way that for all cube's faces, the sum of their edge numbers give the same value.

The cube with its 8 vertices, 6 faces and 12 edges probably is the most well-known polyhedron. The edges can be numbered with integers 1, 2,…,12, whereby numbering may happen in different ways. F.i., numbering can be "central symmetric" or "mirror symmetric". In the first case that means: the numbers for edges arranged symmetrically to the cube's midpoint, for he second case numbers of edges arranged mirror inverted to each other, always add up to 13.
Now, how can (additionally) a numbering be created such that the four edge numbers for any of the cube's faces always have the same sum ?
Because the mean value of all edge numbers is 6.5, the sum of all edge numbers for any face must be 4 x 6.5 = 26.
The cube may be positioned in such a way, that the front top edge carries no. 1 and the left top edge has a smaller number than the edge right above. This allows 40 different enumerations with edge sum 26 for every face, but none of them is central- or mirror-symmetric.
But the numbering of Fig.1 has the property, that the square sums of edge numbers for opposite faces are always equal, in addition it is symmetrically situated to the straight line through the midpoints of top and bottom face.

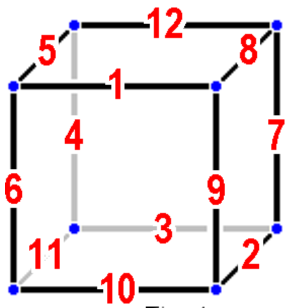


Fig. 1

Any numbering of edges (or of vertices, too) for a certain polyhedron, where the sum of the numbers has the same value for every face, shall be called "magic", as for magic squares, where the sums of row-, column- and diagonal-entries are always equal..

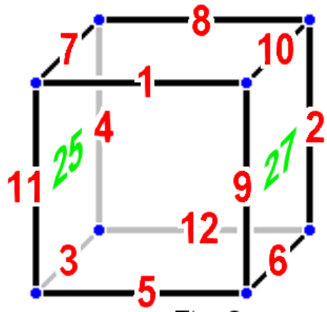


Fig. 2

A magic edge numbering, central-symmetric to the cube's midpoint, turns out to be only almost achievable (Fig. 2).
When, on the other hand, the sums of the three corresponding edge numbers for every of the eight cube vertices should have the same value, there is a problem with integers: the edge sum for every edge would have to be 3 x 6.5 = 19.5 !

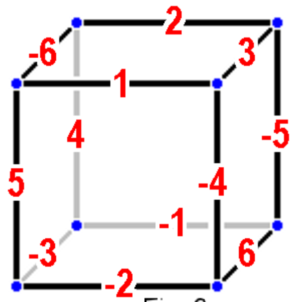


Fig. 3

Another, new way, of numbering should be considered.
The idea, not only to use the integers 1, 2,…,12, but also to admit negative integers f.i. using the 12 integers ± 1, ± 2, …, ± 6, turns out to be successful.. The edge sums of central-symmetric opposite edges, for the eight vertices as well as for the six faces in Fig. 3 are always zero. If now 7 is added to each edge number, there results a numbering with integers 1, 2,…, 13, without 7, with properties:

edge sums for opposite edges: :	each 2 x 7 = 14
edge sums for eight vertices:	each 3 x 7 = 21
edge sums for six faces::	each 4 x 7 = 28.

What now applies to magic numberings of cube vertices with numbers 1, 2, …, 8 ?
The vertex sum for each face must be 4 x 4.5 = 18.

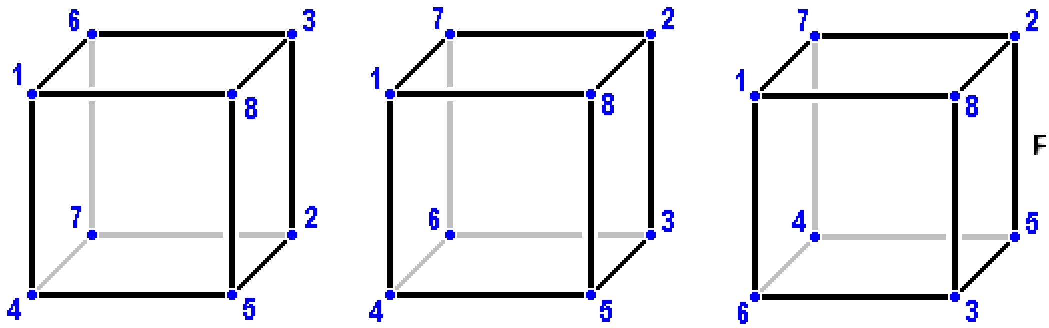
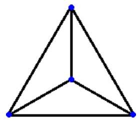


Fig. 4

Here, there exist three essentially different solutions, each of them is mirror-symmetric, in addition the square sums of the right and left cube faces vertex numbers are equal (Fig. 4).
For polyhedra, whose faces all have an even number of vertices, and for polyhedra with odd number of edges resp. vertices it seems worth looking for magic edge or vertex numberings. In the first case the above mentioned integer-problem does not exist, neither in the second case, because the average edge- resp. vertex number is an integer.

For other polyhedra, an experiment with numbers symmetrical to the number 0, as in Fig. 3, comes into question. The simplest polyhedron is the tetrahedron with 4 vertices, 4 faces and 6 edges, here the "magic" condition for edge or vertex numberings requires that two different edges or vertices have the same number, so there is no solution for the tetrahedron.



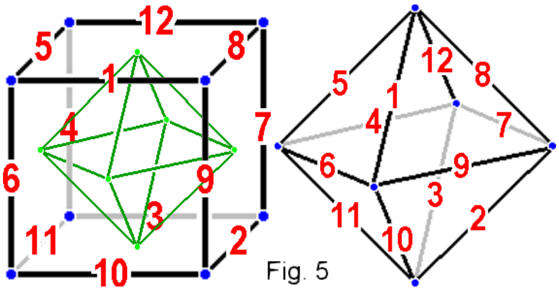


Fig. 5

For the next platonic solid, the octahedron, one can refer to the cube, because these solids are dual to each other, i.e. they result – roughly spoken – from exchanging vertices <=> faces. The magic numberings of Fig. 1 to Fig. 4 can be transferred to the octahedron, we show this with the help of Fig 1; the face sum of 26 for the cube becomes the vertex sum of 26 for the octahedron.
There are two more platonic solids: dodecahedron and icosahedron, dual to each other, therefore one can limit oneself to the dodecahedron with 20 vertices, 12 faces and 30 edges.
Since all of its faces are pentagons, 3 edges meet in each vertex, and its number of edges and vertices is even, there is no possibility for a magic numbering that starts with 1 and runs consecutively; one has to rely on sets of numbers that are symmetrical to the number 0.
An edge numbering succeeds with $\pm1, \ldots, \pm16$ leaving out ±8 and a vertex numbering with the numbers $\pm1, \ldots, \pm11$, where ±6 does not occur; the solutions are both central-symmetric.

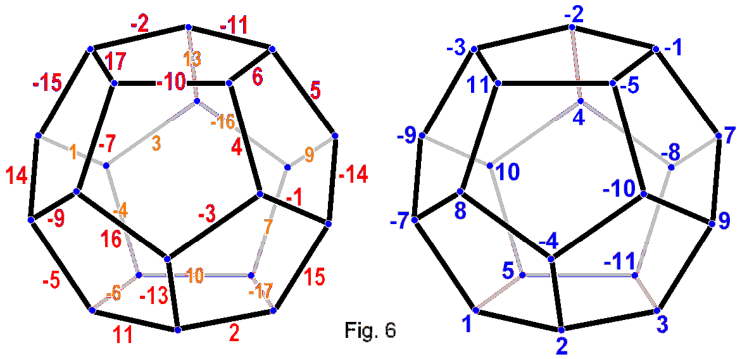


Fig. 6

There are different types of magic numberings for polyhedra, which can be described by specific character sequences. A first letter E, F or K indicates whether vertices (germ.: “Ecken”), faces or edges (germ.: “Kanten”) are numbered, a second (and possibly a third) letter E or F says whether vertex or face sums should be constant An appended letter C indicates that the numbering is “classic”, i.e. consecutive, starting with 1; an appended letter S is intended to indicate that the numbering is central- or mirror symmetric.
Thus: Fig. 1 => KFC, Fig. 2 => KFCS-nearly, Fig. 3 => KFS, Fig. 4 => EFCS, octahedron in Fig. 5 => KEC, Fig. 6 left => KFES, Fig. 6 right => EFS.
A solution of type KFE requires E+F conditions for K = E + F – 2 variables following Euler’s polyhedron formula, and therefore will not exist for every convex polyhedron.
The rhombic dodecahedron with 14 vertices, 12 rhombic faces and 24 edges, allows a numbering

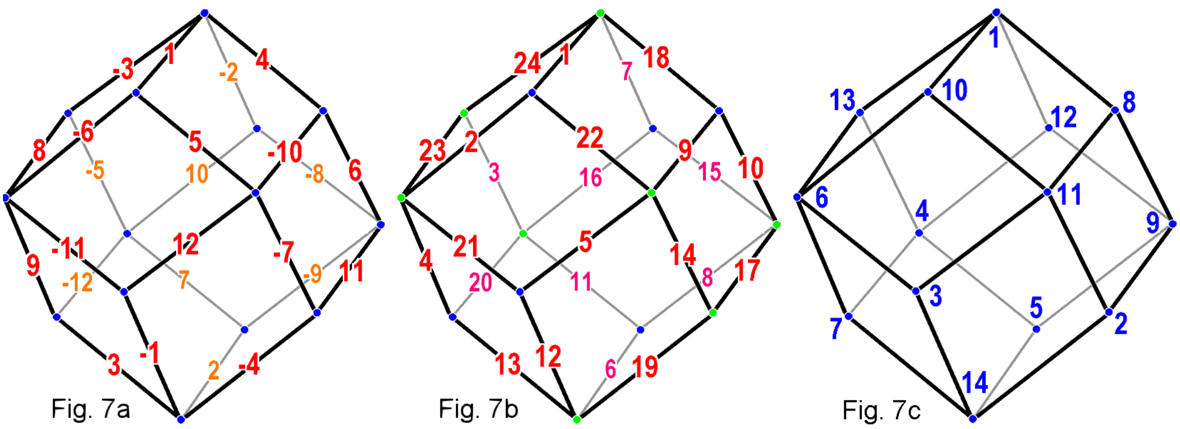


Fig. 7a Fig. 7b Fig. 7c

of type KFE (Fig. 7a) using $\pm1, \pm2, \ldots, \pm12$, but with a rather complex symmetry.
The classic numbering KFCS in Fig. 7b is mirror symmetric; here corresponding edges have sum 25 and all faces, all six 4-valent vertices as well as two of the eight 3-valent vertices (green) have edge sum 50. In the central symmetric Fig. 7c of type EFCS opposite vertices have sum 15 and all faces the vertex sum 30 (there are 12 such solutions).

n-sided Prisms have 2n vertices, n+2 faces and 3n edges. For odd n the average edge number is (3n+1)/2, an integer; numberings of type KFC are possible.

However solutions of types KEC and EFC are only nearly (germ.: “fast”) reachable. In Fig. 8a triangles resp. squares have edge sum 15 resp. 20, in addition bottom and top triangle have equal square sums.

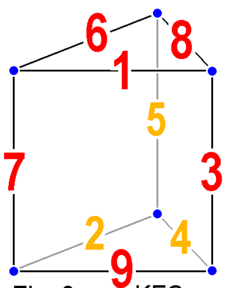


Fig. 8a KFC

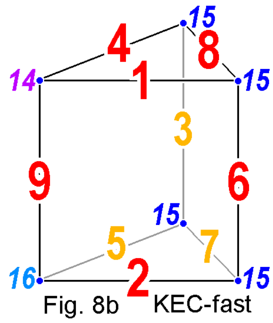


Fig. 8b KEC-fast

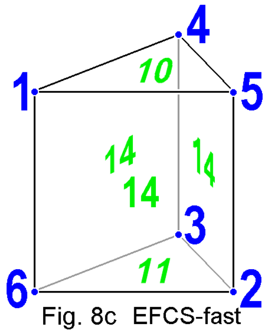
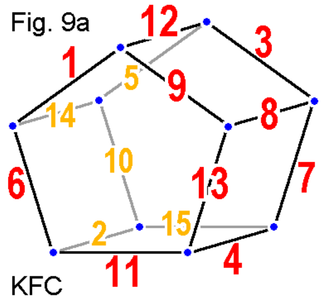


Fig. 8c EFCS-fast

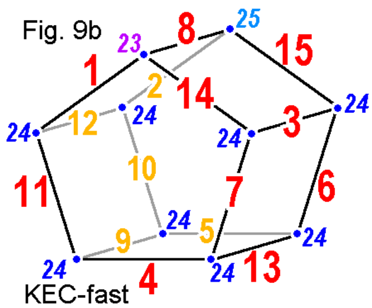
The same applies to the 5-sided prism: In Fig. 9a the edge square sums of the front and the back pentagon are equal.

Fig. 9a



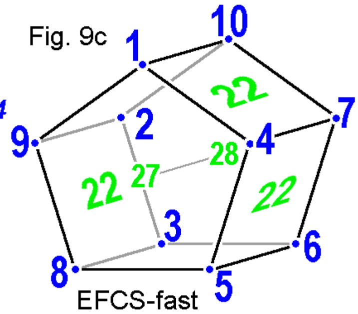
KFC

Fig. 9b



KEC-fast

Fig. 9c



EFCS-fast

Representations for the 7-sided prism are made using Schlegel-diagrams:

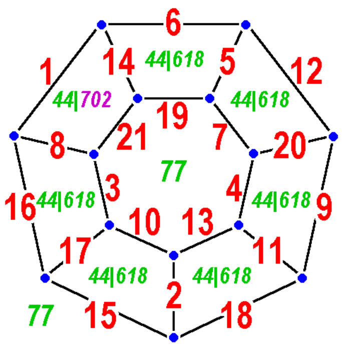


Fig. 10a KFC

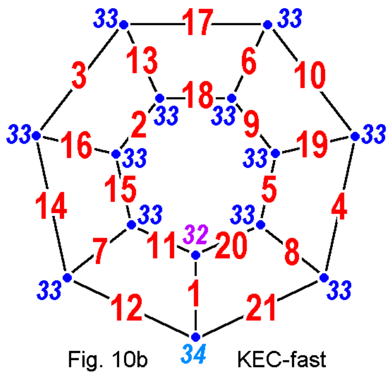


Fig. 10b KEC-fast

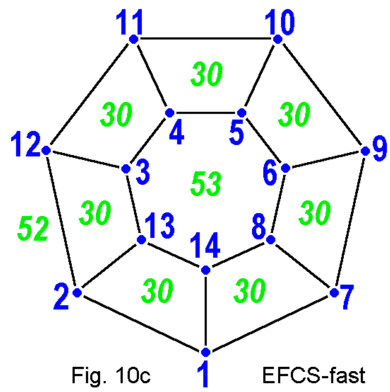
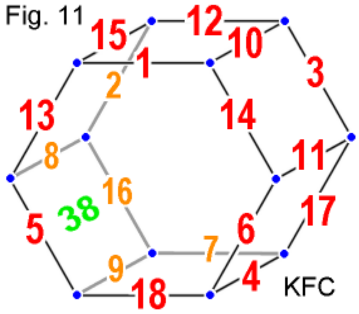


Fig. 10c EFCS-fast

In Fig. 10a one has 21 variables for edge numbers, but first only 9 conditions concerning edge sums of the faces. In this way it is possible to construct a solution, with additionally 6 of the seven quadrilaterals having 618 as edge number square sum. 702 is the closest approximation to 618 here

Fig. 11



KFC

All quadrilateral faces of the 6-sided prism (Fig. 11) have the edge sum 2 x 19 = 38, both hexagons have edge sum 3 x 19 = 57.
The numbering of type KFC is symmetrical to the straight line through the hexagon’s midpoints with the additional property that the square sums of the edge numbers of the hexagons are equal.
Except for rotations and reflections there is exactly one such solution

For the square antiprism (Fig. 12) with 8 vertices, 10 faces and 16 edges is 8.5 the mean sum of edge numbers.

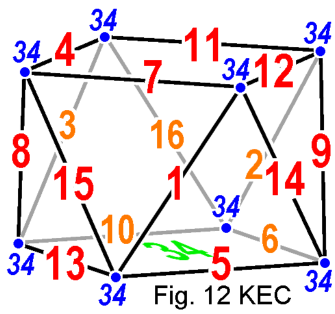


Fig. 12 KEC

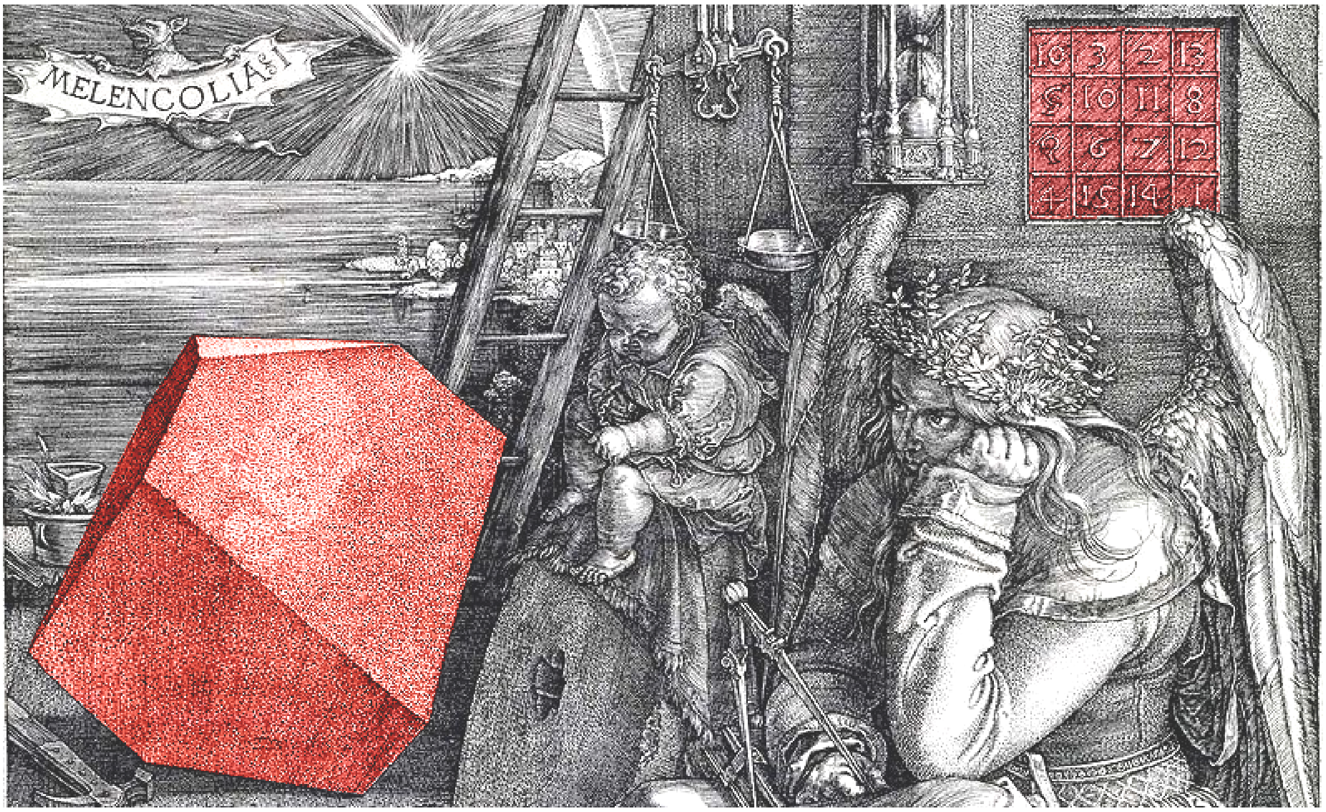
Because 4 edges meet in every vertex, it is worth to look for a type KEC numbering. This requires eight times the edge sum 34. Moreover the top and bottom square should also have edge sum 34, and numbers of opposite triangle edges the sum 17.

This configuration is very reminiscent of a classic 4x4 magic square. There, too, the sum 34 occurs ten times (4 rows, 4 columns and 2 diagonal sums), each formed from four of the numbers 1, ..., 16.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

A famous magic 4x4 square can be found (colored red) in the picture "Melencolia" by Albrecht Dürer (image section below). In the fourth row of this square are the numbers 15 and 14; The year 1514 is the year of the death of Dürer's mother.

A polyhedron is also shown on the Melencolia image, colored red.



This solid has 12 vertices, 8 faces (2 triangles, 6 pentagons) and 18 edges. Classic edge or vertex numbering is impossible here, but numbering with sets of numbers symmetrical to zero can be considered (Fig. 13).

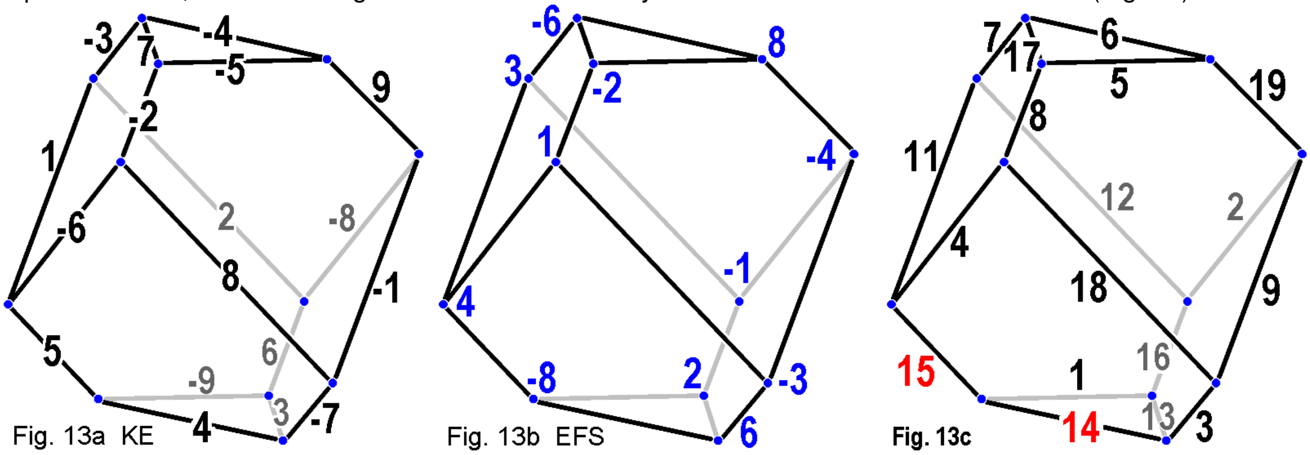


Fig. 13a KE   Fig. 13b EFS   Fig. 13c

In Fig. 13a the edges are numbered with ±1, …, ±9; the 3 edges of any vertex add up to zero.

Fig. 13b shows a central symmetric vertex numbering using the 12 numbers ±1, ±2, ±3, ±4, ±6 und ±8; all eight faces have vertex sum zero..

If 10 is added to each edge number in Fig. 13a, the edge sum is 30 for all twelve vertices (Fig. 13c) and two of the lower edges show 15 and 14.